REVIEWS

BIFURCATIONS AND THE GENERATION OF CHAOS IN A SIMPLE ELECTRIC CIRCUIT

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Laboratory investigation of complicated chaotic regimes in a simple noise generator is described. The investigation includes detailed introduction into the problem, qualitative analysis of the noise generator, and description of the routine for computer processing of results of the experiment.

1. Introduction. Origin of Stochastic Processes in Ordered Systems

Our every-day experience and traditional education lead us to the conclusion that random, complicated, and irregular behavior is a feature of complicated systems. Examples are the disordered motion of molecules in a vessel filled with gas or the behavior of a crowd of infuriated fans when a footfall match has been cancelled suddenly. We are usually unable to relate cause and effect unambiguously in such complicated systems, that is we cannot predict the processes that occur in them in detail and conclude that they are random. Nevertheless, there is always the hope that these random features and this unpredictability could be eliminated if one had more precise knowledge about the system. It was taken for granted for a long time that if complete information was available about the interactions between all components of a complex system and its initial state was completely specified, its behavior
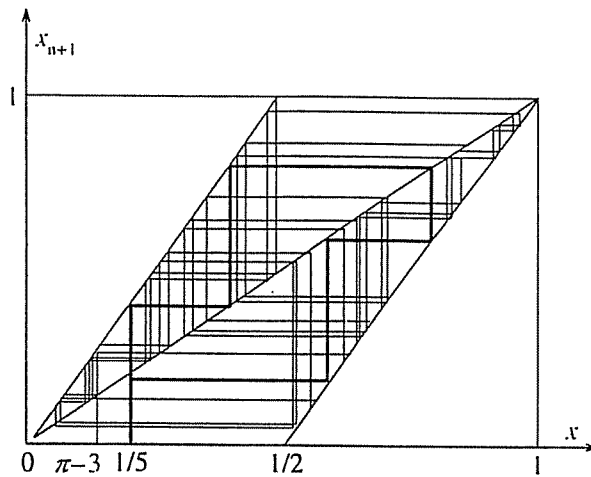
Fig. 1. The periodic (thick line) and random (thin line) trajectories in the form of sequential steps for the simple system $x_{n+1} = \{2x_n\}$ plotted by a personal computer.

dynamic chaos is organized. Now we have to answer a natural question; namely, if there are periodic systems among random ones, is it possible that only periodic systems are encountered in reality?

In fact, periodic systems are not stable, and given a small error in x_1 we get a fundamentally different sequence (see Fig. 1). This demonstrates another feature of dynamic chaos, namely its sensitivity to changes in the initial conditions. Thus, although there is an infinite number of periodic sequences (but even more aperiodic sequences), it is practically impossible to realize them all. This is because they form an almost continuous set and are in a way indistinguishable. Suppose that we have a small error in the initial value then we don't get that sequence, but the one we do get still belongs to the set of unstable trajectories. Thus one of them is always observed. This leads us to the interesting conclusion that although a particular trajectory of our stochastic set cannot be observed or realized because of instability, the set as a whole is stable, and one of its trajectories is observed.

Why was random behavior of nonrandom systems never seen before? Could it be that our ideas are only applicable to artificial systems, and life is different, perhaps real chaotic systems are described by different equations? Could Newton's equations guarantee the regular behavior of mechanical systems?

Let us consider a very simple example, namely the motion of a swing. It is certainly described by the ordinary equations of mechanics, but even this example is flawed. We know that the swing moves periodically, but in fact the period is determined by the rider, who can vary the effective length of the swing by sitting and standing, thus accelerating the seat or retarding it by moving out of phase. Let us deprive the rider of the freedom of choice and replace him by a mechanical man that sits and stands periodically. The swing will oscillate irregularly without any external force. By sitting at a constant frequency that is out of phase with the swing, the man will lengthen or shorten the pendulum and thus supply energy to the swing or take it away. The process will be random because at different angles the swing will move at different velocity. For example, when the seat is at the low position the velocity is zero or very small. Thus a real mechanical system moving according to Newton's laws behaves chaotically, i.e., it generates chaos. If you ask why this was not seen before, we think that in fact it was seen, but not recognized. Within the traditional framework isolated experiments on the chaotic behavior of simple systems could be neglected. In some way or other, randomness was interpreted in terms of 'unaccountable' noise or fluctuation.

Now we proceed to a more accurate description of dynamic systems which display chaotic behavior (sequences can be interpreted as dynamic systems with discrete time). The phase space of such systems

can be found very easily. It follows from the formula $x_{n+1} = F(x_n)$ that after one mapping the initial probability density $\rho_j(x)$ defined on the closed interval transforms to

$$\rho_{j+1}(F(x)) = \sum P_j(x)|dF(x)/dx|^{-1},\qquad(2)$$

where the sum is taken over all branches of the function $F(x)$. This equation can be explained as follows: the initial distribution becomes less dense by the factor dF/dx (the mapping is stretching), but after several operations points from different parts of the initial interval are mapped to the interval dx because the mapping is not one-to-one. Mappings like that in Fig. 1 have an invariant probability distribution $P(x)$, which can clearly be derived from the condition $\rho_{j+1} = \rho_j = P$, i.e., $P(x)$ should satisfy the equation

$$P(F(X)) = \sum P(x)|dF(x)/dx|^{-1}.\qquad(3)$$

By using a direct substitution into Eq. (3) we can prove that $P(x) = $ const for piecewise linear mappings like $x_{n+1} = \{2x_n\}$. Since the integral of the probability is unity, we can set $P = 1$ and hence for the mapping $\langle x \rangle = \int_0^1 x dx = 1/2$, the variance is $D = \langle (x - \langle x \rangle)^2 \rangle = 1/12$, and the correlation function [3] is

$$K(j) = D^{-1}\langle [(x_1 - \langle x \rangle)(x_{j+1} - \langle x \rangle)] \rangle = 12 \int_0^1 (x - 1/2)(\{2^j\} - 1/2)dx = \exp[-(\ln 2)j].$$

We can see that in our case the correlations fall exponentially with time. The exponent characterizing the rate of fall of the correlation and the rate of spread of the trajectories, i.e., Lyapunov exponent, is the Kolmogorov–Sinai entropy. In this case the entropy is $h = \ln 2$.

Is stochastic behavior possible in systems that are not described by discontinuous mappings like the one in Fig. 1 but by smooth ones? The answer is yes, but not in all cases.

Consider the set of mappings $x_{k+1} = F(x_k)$ which depend on a parameter b

$$x_{k+1} = bx_k(1 - x_k).\qquad(4)$$

At $b = 4$ the maximum $x = 1/2$ is a pre-image of the stationary unstable point $x = 0$ (the point $x = 0$ follows $x = 1/2$). After the variable change $y = \varphi(x) = (2/\pi)\arcsin\sqrt{x}$ [4], the mapping defined in (4) at $b = 4$ becomes the piecewise linear mapping

$$F(y) = \begin{cases} 2y, & 0 \leq y \leq 1/2, \\ 2(1 - y), & 1/2 \leq y \leq 1. \end{cases}\qquad(5)$$

We have already proved that this mapping has an invariant probability distribution. Hence at $b = 4$ the mapping defined in (4) also has the invariant probability distribution $\left[\pi\sqrt{x(1 - x)}\right]^{-1}$.

2. Noise Generator. Approximate Description and Experiment

When investigating stochastic processes in dynamic systems by the methods of the theory of oscillations we have to construct the stochastic set, to understand the mechanism by which chaos is generated, to formulate its criteria, and then, after finding a small parameter in the system, we obtain an approximate description of the system's behavior in the stochastic region. This procedure is only possible for relatively simple systems with a three-dimensional phase space that can be described by two-dimensional or, approximately, three-dimensional Poincaré mappings. Consider a simple electronic generator of stochastic oscillations.
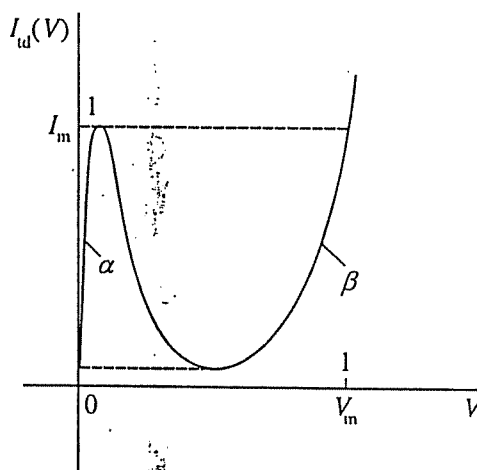
Fig. 4. Current–voltage characteristic of the tunnel diode used in the noise generator.

high-voltage branch. When $R < 11\ \Omega$, the triode nonlinearity can be ignored, and the signal consists of trains of exponentially rising oscillations, while switching from one train to the other is accompanied by a voltage pulse across the diode. When $R < 11\ \Omega$ no periodic oscillation are detected, and in any case a random signal with a continuous spectrum is generated. We can see in the spectra and oscillograms shown in Fig. 5 that the growth rate of the oscillation amplitude h rises as the resistance in the circuit falls, while the average train duration gets shorter, and the peaks in the spectrum, corresponding to the train repetition frequency, get broader. Most of the power is contained in the main peak, corresponding to the circuit's natural frequency.

We can use dimensionless variables in Eq. (6) by the substitution $x = I/I_m$, $z = V/V_m$, $y = UC^{1/2}/(I_m L^{1/2})$, $\tau = t(LC)^{1/2}$. We then have

$$\dot{x} = 2hz + y_q z, \quad \dot{y} = -x, \quad \mu\dot{z} = x - f(z),$$

where $h = 0.5(MC - rC)(lc)^{1/2}$ is the growth rate of the oscillations in the circuit without the diode, $g = V_m C^{1/2}/(I_m L^{1/2})$ is the parameter characterizing the effect of the tunnelling diode, $\mu = gC_1/C \ll 1$ is a small parameter proportional to the diode capacitance, and $f(z) = I_{td}(V_m z)/I_m$ is the dimensionless characteristic of the diode (Fig. 4).

The system (7) includes a small factor μ multiplied by a derivative, hence motion in phase space may be separated into fast (diode switching at $x = $ const and $y = $ const) and slow motion, in which the voltage across the diode follows the current [the trajectories belong to the surfaces A ($x = 0$) and B ($x = f(z)$, $F'(z) > 0$) corresponding to the α and β branches of the diode characteristic].

The system has one unstable (at $2h > g/f'(0)$) equilibrium at the saddle point $x = y = z = 0$. The trajectories on the surface A wind around the unstable focus and, in the long run, reach the surface B. From here the system moves along a fast trajectory to the bottom of the surface B. After moving across B, the system switches along a fast trajectory back to surface A to the neighborhood of the equilibrium point, and a new train of rising oscillations is generated. This pattern corresponds to the real trajectories in oscillograms of Fig. 5.

3. Statistical Description of the Simple Noise Generator

Now we shall demonstrate that at $\mu = 0$ the noise generator can be also described in terms of a multivalent mapping of an interval to itself. The mapping is, however, more complex than the one in
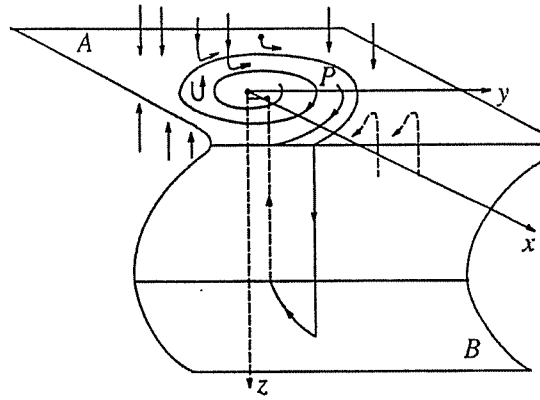
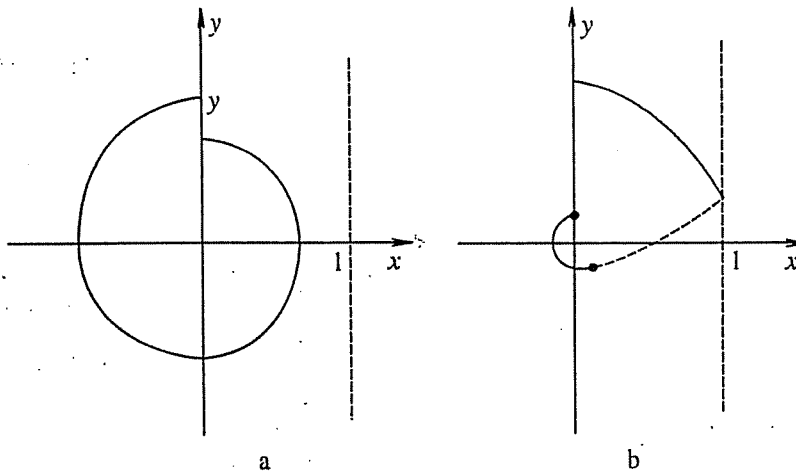**Fig. 6.** Phase space of the system described by Eq. (7).

Fig. 7. Poincaré mapping for the system (9): (a) the trajectory belongs to one surface of slow motion; (b) the trajectory rapidly moves to the second slow-motion surface and returns back.

The mapping consists of two parts. The function $F_1(y_j)$ describes that part of the mapping represented by trajectories not extending to the half-plane B (Fig. 7a), and the function $F_2(y_j)$ describes the part represented by trajectories belonging to both half-planes (Fig. 7b). From Eq. (9) we immediately obtain

$$Y_{j+1} = F_1(y_j) = \exp(2\pi\nu)y_j \equiv ky_j. \tag{10}$$

The function $F_2(y_j)$ cannot be explicitly derived from Eq. (9b), and therefore we approximate it with a formula which qualitatively describes the trajectory's stochastic behavior, i.e.,

$$y_{j+1} = F_2(y_j) = y_0 - (y_j - y_0)^{1/2}. \tag{11}$$

Thus, when $y_j < y_0$ we use the branch of the mapping defined in (10), and when $y_j > y_0$ the branch defined in (11). The 1/2 exponent in Eq.(11) reflects the fact that the trajectories approach the line of fast motion $x = 1$ approximately tangentially. The constant y_0 is the shift of the trajectories when moving on plane B. By combining (10) and (11), we obtain the mapping $y_{j+1} = F(y_j)$, which is shown in Fig. 8. This mapping has an attractive area, i.e., the attractor $y_0 - ky_0 - y_0)^{1/2} < y < ky_0$. If $0 < k - 1 < (4y_0)^{-1}$, then the mapping inside the attractor is stretching, i.e., $|dy_{j+1}/dy_j| > 1$.
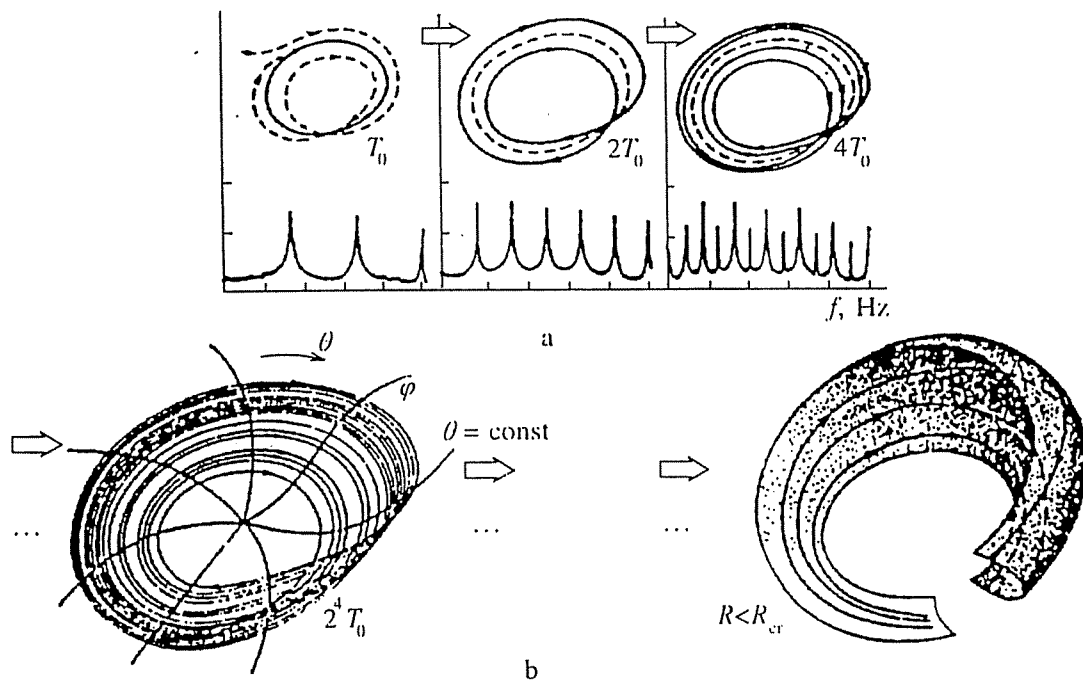
Fig. 10. Generation of a strange attractor in a three-dimensional system by sequential period-doubling bifurcations (initially the motion period is T_0): (a) sequence of doublings in the phase space (top) and in spectra (bottom); (b) the strange attractor in the shape of a folded and closed strip which is generated when the motion with period $2^n T_0$ becomes stochastic (the band section has Cantor structure).

The motion of many three-dimensional systems can be described in terms of this mapping. Take, for instance, the system whose attractor is a closed widening strip with a fold (Fig. 10). The coordinate φ on the section is transformed using the mapping in (4).

This mapping has a stationary point $x_{k+1} = x_k = x^* = 0$ at all b, and at $b > 1$ there is a second point $x^* = 1 - 1/b$. This point is stable until $b = 3$. For $b > 3$ this nontrivial stationary point is unstable because the multiplier dx_{k+1}/dx_k becomes less than -1 at $b = 3$, and there is a stable motion with a period of 2. After this transition the equation $x_{k+1} = x_k$ has two roots. The nondegenerate stationary point does not disappear, but becomes unstable. The double cycle is stable in the interval $3 < b < 3.45$. At $b \approx 3.45$ the double cycle becomes unstable, and a stable four-fold cycle is generated. At a higher b this cycle also becomes unstable, and a stable cycle with a period of 2^3 is generated, then $2^4, \ldots, 2^n, 4^{n+1}$, and so on. Finally, at $b_\infty \approx 3.57$ no stable periodic motion is possible, and the system is completely randomized. On this transition, a strange attractor is generated in the three-dimensional phase space (Fig. 10). Note, that even for $b > 3.57$ stable periodic points may exist; for instance, at $b = 3.83$ there is a stable three-fold cycle [7].

Figures 11 and 12 show the results of a real experiment with the noise generator of Fig. 3. The sequence of period-doubling bifurcations and the resulting transition to the chaotic behavior are clearly seen. A computer simulation using Eqs. (7) and (8) produces similar results as can be seen in Figs. 13 and 14.

5. Computerized Simulations

The analog signal generated by a stochastic generator is a continuous function of some physical variable (such as voltage) versus time $x(t)$. For a computer simulation, the signal has to be a set of values of the
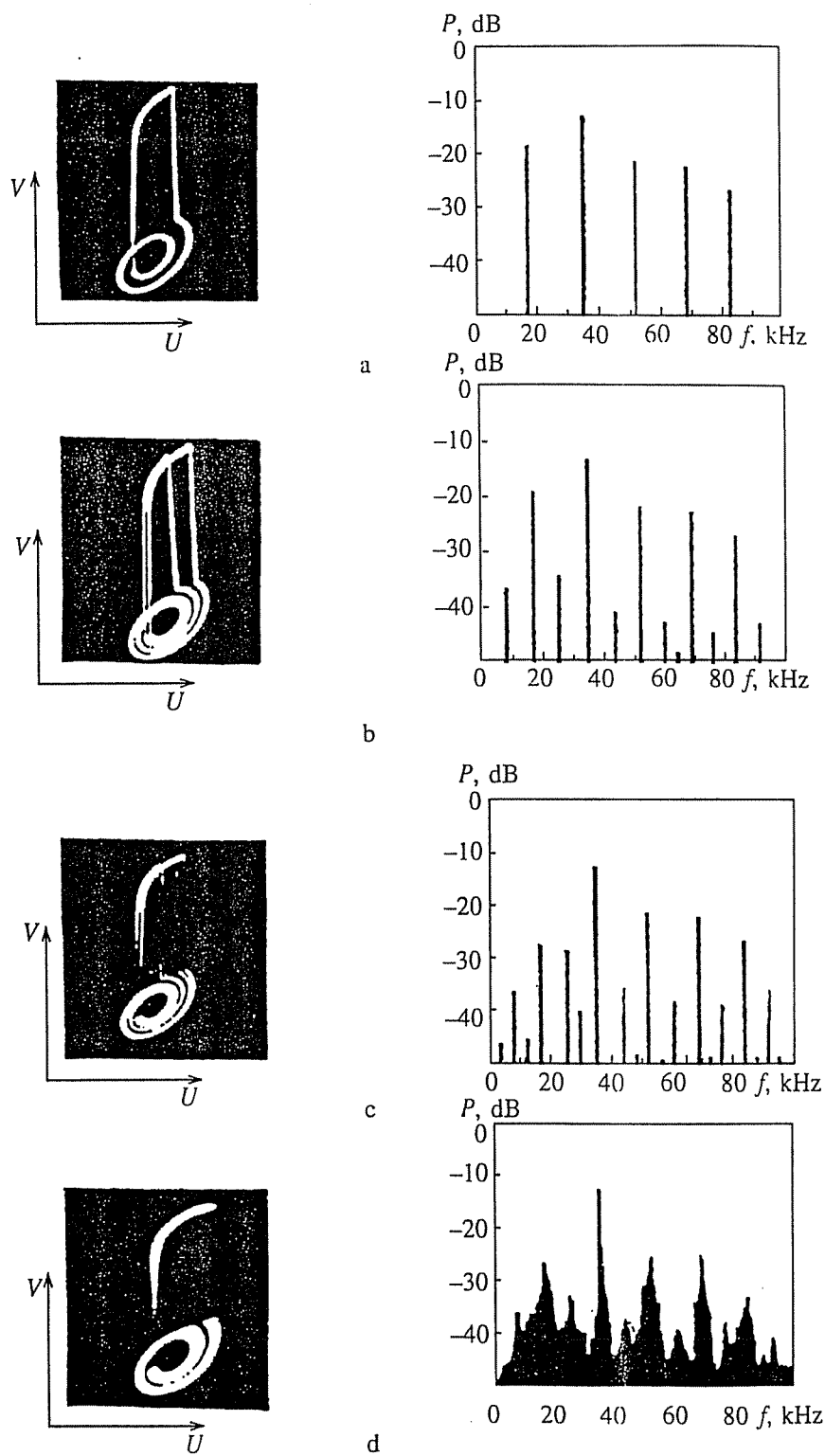
Fig. 12. Evolution of phase portraits and power spectra recorded in a real experiment [8]: (a) 1:1; (b) 2:2; (c) 4:4; (d) chaos.
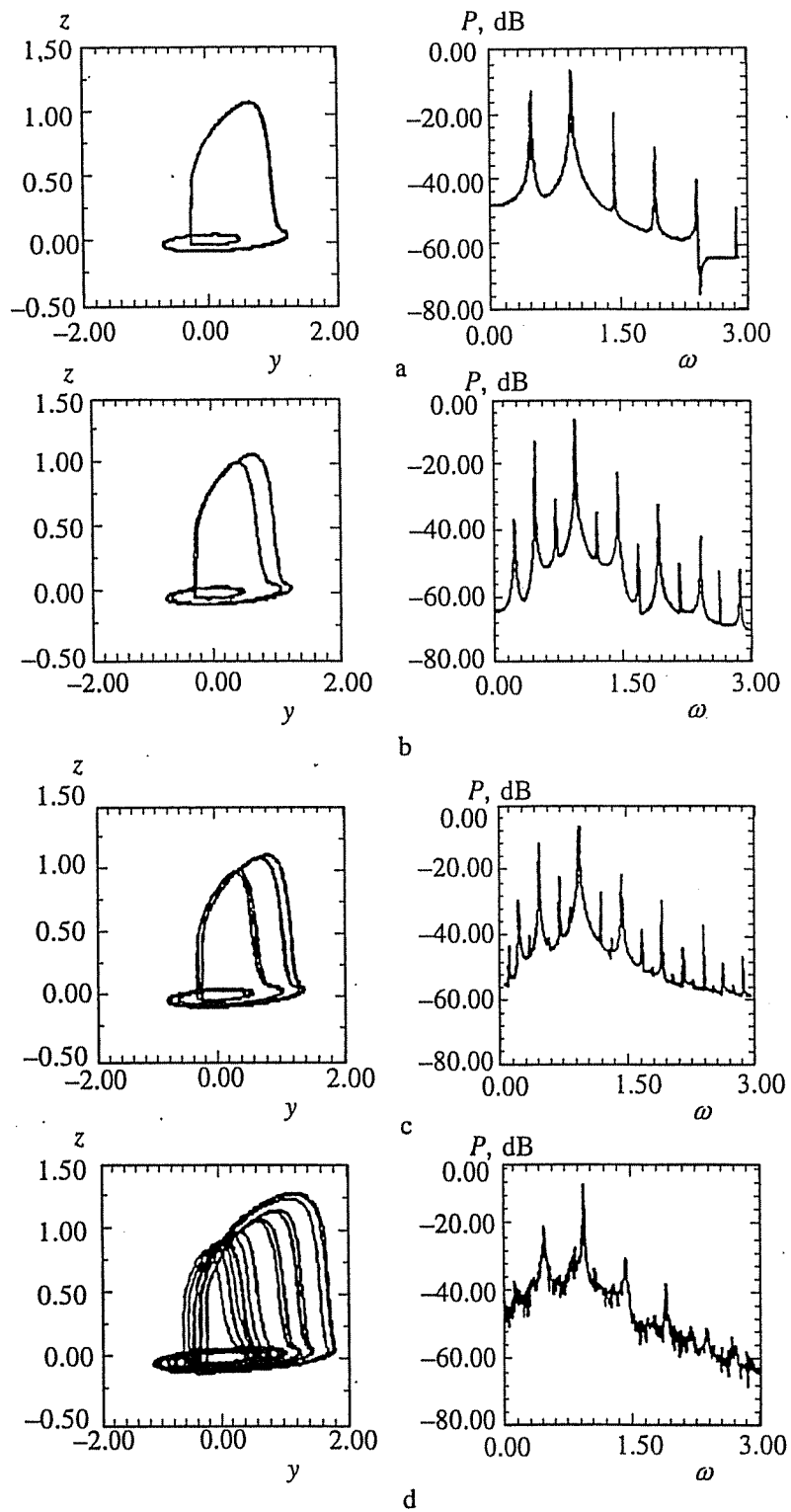
Fig. 14. Phase portraits and power spectra for system (7), (8) (computer simulation) [8]: (a) $\delta = 0.060$ (1:1); (b) $\delta = -0.067$ (2:2); (c) $\delta = -0.07$ (4:4); (d) $\delta = -0.076$ (chaos).

5.2. Function of Mutual Information

An important parameter in the analysis of the sequence $\{x_j\}$ is the time interval ΔT after which the prehistory is completely forgotten by the system's 'memory.' In other words, the state of the system at t_0 only depends on its states within the time interval $[t_0 - \Delta t, t_0)$ and is not affected by what happened before $t_0 - \Delta T$. For periodic and quasiperiodic signals $\Delta T \to \infty$, but for a stochastic signal the quantity ΔT is finite. The limiting case of the stochastic signal is white noise, for which $\Delta T = 0$. A white-noise signal at any moment can be treated as 'new'. The time ΔT (or $\overline{m} = \Delta T / \Delta t$ for a discrete sequence) can be estimated by the shape of the autocorrelation function. Another approach is to construct the mutual information function

$$I(m) \sim \ln \left\langle \frac{P_{k,k+m}(x_i, x_j)}{P_k(x_i)P_{k+m}(x_j)} \right\rangle_{i,j,k}, \tag{13}$$

where $P_k(x_i)$ is the probability that x_i can be described at the k-th moment of time, and $P_{k,k+m}(x_i, x_j)$ is the joint probability of detecting x_i at the k-th moment and x_j at the $(k + m)$-th moment. The function is averaged over all i, j, and k from the set $\{1, 2, \ldots, n\}$. In practice the mutual information function can be calculated by the formula

$$I(m) = \sum_{i,j=1}^{n} P_{k,k+m}(x_i, x_j) \ln \frac{P_{k,k+m}(x_i, x_j)}{P_k(x_i)P_{k+m}(x_j)}. \tag{14}$$

The correlation parameter $\overline{m}$ can either be derived from the position of the first maximum of $I(m)$, if it has been detected, or from the interval of m over which the function $I(m)$ drops by a factor of e. Note that the presence of a maximum on the $I(m)$ curve is related to the property of a strange attractor that after a certain time interval any trajectory returns to the ε-neighborhood of the point of its origin.

5.3. Correlation Dimension

Consider again the time sequence $\{x_j\}$, $j = 1, \ldots, n$. Following Takens' procedure, we can reconstruct the trajectory in a p-dimensional pseudo-phase space using as coordinates x_j, x_{j+m}, $x_{j+2m}, \ldots, x_{j+(p-1)m}$, where m is a time delay selected in a proper way. As a result, by changing $j = 1, \ldots, n - pm$ we obtain a set of p-dimensional vectors that describe the phase trajectory of the dynamic system. We can prove that if the trajectory is closed in the ordinary phase space, it is also closed in pseudo-phase space, and if the trajectory is chaotic in phase space, it remains chaotic in the pseudo-phase space. A more general statement reads: all the basic properties of the attractor to which the trajectory belongs in phase space are conserved by a transformation to pseudo-phase space.

The positions of two points belonging to the same trajectory, but separated in time, are uncorrelated in the chaotic situation corresponding to a strange attractor in phase space and, hence, in pseudo-phase space. Since all the points belong to the attractor, there is a spatial correlation, which can be characterized by some function. The correlation function is defined as

$$C(r) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i,j=1}^{n} H(r - \|\mathbf{x}_i - \mathbf{x}_j\|), \tag{15}$$

where H is the Heaviside function (defined to be equal to unity for positive arguments and to zero for all the other arguments), $\|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between the two points $\mathbf{x}_i = (x_i, x_{i+m}, \ldots, x_{i+(p-1)m})$ and $\mathbf{x}_j = (x_j, x_{j+m}, \ldots, x_{j+(p-1)m})$ in the p-dimensional pseudo-phase space. The number of points in Eq. (15) with distances between them less than r can be calculated. For many attractors the function $C(r)$ become a power function as $r \to 0$, i.e.,

$$\lim_{r \to 0} C(r) \sim r^d, \tag{16}$$

(ii) the number of points n_i to be displayed; the first $\min(n_1, n)$ points are displayed, where n is the total number of points in the file;

(iii) the format of the output data is one column of floating point numbers or one column of integers.

The routine `plott.exe` creates three files `main.dat` (if the initial data file includes integers), `power.in`, and `minfo.in`. The latter two files contain initial data for the routines `power.exe` and `minfo.exe`. The file `main.dat` contains floating point numbers. Other subroutines process the data in this format and request the file `main.dat`. If the initial file `calibr.dat` contains floating point numbers, the file `main.dat` is not created and all the subroutines process the initial data file.

2. The routine `power.exe` creates the power spectrum. It takes as input the data in power.in. The resulting file power.out contains the amplitudes of the first 65 components of the power spectrum. The total spectrum includes 128 harmonics, but the harmonics with numbers higher than 65 are identical to the corresponding lower harmonics since the spectrum is symmetrical.

3. The routine `plotp.exe` reads `power.out` file and displays it on the monitor.

4. The routine `minfo.exe` calculates the mutual information function. It uses the fifty closest points. The distance $\overline{m}$ over which the prehistory of the system is lost is found from the first maximum of the function. Since the correlation dimension is calculated from the same data file as the mutual information function, it is desirable that $\overline{m} = 2 - 3$. If $\overline{m} = 1$ ($\overline{m} > 3$), the initial data were measured with too large (small) a quantization step in time. In the first case, the data are not reliable, and in the second case excessive. The initial data for `minfo.exe` are in `minfo.in`. The program creates the file `minfo.out` with the mutual information plotted against $m = 1, \ldots, 50$.

5. The routine `plot.exe` graphically displays the mutual information function on the monitor.

The second part of the package calculates the correlation dimension and K_2-entropy and is started by the batch file `2.bat`. But, before running it, the file `dim-par.in` has to be input with the following data:

(a) name of the data file with the signal in floating point numbers (after running `1.bat`, this file exists and its name is recorded in `minfo.in` and `power.in` files);

(b) the number of points in the input file n (this information is contained in `minfo.in` and `power.in`);

(c) the number of reference points n_{ref} (it is recommended that $n_{ref} > 200$);

(d) the initial dimension of the embedding space p_0 (the correlation functions are calculated in embedding spaces with dimension p: $p_0 \leq p \leq p_0 + 8$, a value of $p_0 = 2$ is recommended);

(e) the delay m (derived from the first maximum of the mutual information function and contained in `minfo.out`).

The file `2.bat` sequentially runs the following two routines:

(i) `dim-gri.exe`, which reads data from dim-par.in and the initial data file, graphically displays the attractor reconstructed in the three-dimensional pseudo-phase space, and calculates the correlation functions for $p_0 \leq p \leq p_0 + 8$. The functions are written to `output.dat`.

(ii) `dim-gr2.exe`, which reads data from `output.dat` and displays $\ln C_p(r)$ versus $\ln r$ and $d \ln C_p(r)/d \ln r$ versus $\ln r$ (local slope) at $p_0 \leq p \leq p_0 + 8$.

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