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THE INTERACTION OF STRUCTURES AND SPATIOTEMPORAL CHAOS IN MODELS OF COUPLED WAVES

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Dynamic and stochastic effects of the interaction of individual localized structures and lattices of structures of a nonlinear field are considered within one-dimensional large-box Ginzburg–Landau and Swift–Hohenberg models. The criterion of soliton "survival" in competition with counterpropagating modes is derived analytically for a quasigradient case. The criterion of linear stability of a spatially homogeneous regime, that is similar to the Benjamin–Feir condition, is obtained for coupled Ginzburg–Landau equations. The relation between the correlation dimensions of the space series of counterpropagating modes and the dimension of the time series is investigated. It is shown that within a quasigradient model of two weakly coupled counterpropagating modes, the field can be represented as a superposition of two fixed stochastic spatial lattices which correspond to these modes and move through one another at constant velocity. The dimension of the time series at a given point in space is close to the sum of dimensions of the spatial distributions of counterpropagating modes. With the increase of coupling, the interaction of coupled modes "loosens" the equilibrium state corresponding to fixed lattices and the dimension of the time series grows.

1. Introduction

The purpose of this paper is the analysis of the interaction of localized structures of a nonlinear field within one-dimensional Ginzburg–Landau and Swift–Hohenberg models.

The first two models considered in the paper refer to the class of nearly gradient systems and are described by the following equations:

$$\partial_t u_1 + v\partial_x u_1 = -u_1 + \beta_1 u_1 |u_1|^2 - u_1 |u_1|^4 - (k_{01}^2 + \partial_x^2)^2 u_1 - \varepsilon_1 u_1 |u_2|^2 , \quad (1)$$

$$\partial_t u_2 - v\partial_x u_2 = -u_2 + \beta_2 u_2 |u_2|^2 - u_2 |u_2|^4 - (k_{02}^2 + \partial_x^2)^2 u_2 - \varepsilon_2 u_2 |u_1|^2 \quad (2)$$

for the generalized Ginzburg–Landau model, and

$$\partial_t u_1 + v\partial_x u_1 = -u_1 + \beta_1 u_1^2 - u_1^3 - (k_{01}^2 + \partial_x^2)^2 u_1 - \varepsilon_1 u_1 u_2^2 , \quad (3)$$

$$\partial_t u_2 - v\partial_x u_2 = -u_2 + \beta_2 u_2^2 - u_2^3 - (k_{02}^2 + \partial_x^2)^2 u_2 - \varepsilon_2 u_2 u_1^2 \quad (4)$$

for the Swift–Hohenberg model. Here β_i is the parameter determining the instability threshold, $1/k_{0i}$ is the characteristic spatial scale of the system, and v is the group velocity. The fields $u_1(x,t)$ and $u_2(x,t)$ correspond to waves travelling to the right and to the left, respectively ($v > 0$).

The third model refers to the class of nearly Hamiltonian systems and is described by equations of the form

$$\partial_t u_1 + v\partial_x u_1 = u_1 + (1+i\alpha_1)\partial_x^2 u_1 - (1+i\beta_1)u_1|u_1|^2 - (l_1+id_1)u_1|u_2|^2\,, \qquad (5)$$

$$\partial_t u_2 - v\partial_x u_2 = u_2 + (1+i\alpha_2)\partial_x^2 u_2 - (1+i\beta_2)u_2|u_2|^2 - (l_2+id_2)u_2|u_1|^2\,. \qquad (6)$$

It is assumed that $|\alpha_i|, |\beta_i| \gg 1$. The parameter v, like before, has a sense of group velocity for the waves travelling in opposite directions.

Equations of the form (1)–(6) have been broadly discussed in the literature. For example, a system of two coupled complex Ginzburg–Landau equations with a flow component $\pm v\partial_x$ of the field were considered by Dangelmahr & Knobloch [1990] who derived asymptotically equations for slowly varying complex amplitudes of the field and then used them for the investigation of the confined travelling waves and the blinking states observed in experiment. Solutions of this type within coupled Ginzburg–Landau equations were also considered by Cross [1988]. Coupled Ginzburg–Landau equations were employed in the description of the oscillatory instability of Rayleigh–Bénard convective rolls [Croquette & Williams, 1989; Janiaud *et al.*, 1992] and in the description of the evolution of gasless combustion fronts [Matkowsky & Vlopert, 1992]. Numerical investigation of the competition of localized structures within coupled generalized Ginzburg–Landau equations (with different higher-order derivatives taken into account) was performed by Brand & Deissler [1989, 1991].

This paper is concerned, primarily, with dynamic and stochastic effects emerging as a result of the interaction of individual localized structures or lattices of structures described by models (1)–(6). The mechanisms responsible for the interaction of individual localized structures within quasigradient models (1)–(4) at different modes: travelling to the right [Eq. (1) or Eq. (3)] and travelling to the left [Eqs. (2)–(4)] are investigated in Sec. 2. The interaction of counterpropagating waves gives rise to the competition of localized structures at these modes: solitons of one mode suppress the structures moving in the opposite direction. This effect governs the dynamics of two counterpropagating lattices of structures, which is analysed in Sec. 3. Analogous dynamics of the interaction of localized structures is considered in Sec. 4 for the quasiconservative model (5)–(6). Finally, of particular interest is the investigation of dimension and entropy of spatial distributions for the field formed by counterpropagating chaotic lattices of localized structures, of the relation between spatial and temporal dimensions for such fields, as well as of the changes in these quantities that are stipulated by the interaction of the structures. These problems are studied in Sec. 5.

2. Localized Structures

We represent Eqs. (1)–(4) in the form

$$\partial_t u_i \pm v\partial_x u_i = -\frac{\delta F_i^{(1)}}{\delta u_i^*} - \varepsilon_i u_i |u_j|^2\,, \qquad (7)$$

where $F_i^{(1)}$ is a free energy functional and can be written as

$$F_i^{(1)} = \int \{|u_i|^2 - \frac{1}{2}\beta_i|u_i|^4 + \frac{1}{3}|u_i|^6 + |(k_{0i}^2 + \partial_x^2)u_i|^2\}dx \qquad (8)$$

for model (1)–(2) and

$$\partial_t u_i \pm v\partial_x u_i = -\frac{\delta F_i^{(2)}}{\delta u_i} - \varepsilon_i u_i u_j^2\,, \qquad (9)$$

where

$$F_i^{(2)} = \int \{\frac{1}{2}u_i^2 - \frac{1}{3}\beta_i u_i^3 + \frac{1}{4}u_i^4 + \frac{1}{2}[(k_{0i}^2 + \partial_x^2)u_i]^2\}dx \qquad (10)$$

for model (3)–(4).

From Eqs. (7), (9) it is apparent that, in the absence of coupling ($\varepsilon_i = 0$), each traveling wave is described by a gradient model. Indeed, passing over to the moving coordinate $\xi_i = x \mp vt$ we have

$$\frac{\partial u_i}{\partial t} = -\frac{\delta F_i^{(1)}}{\delta u_i^*} \qquad (11)$$

for the first case, and

$$\frac{\partial u_i}{\partial t} = -\frac{\delta F_i^{(2)}}{\delta u_i} \qquad (12)$$

for the second case.

Only statical attractors may exist in the phase space of such gradient systems because the functionals $F_i^{(1,2)}$ decrease along the trajectories of the system ($dF_1^{(1,2)}/dt = -\int |\partial u_i/\partial t|^2 d\xi_i < 0$). Field distributions that are homogeneous in space correspond to the simplest attractors. Each of the equations (11) and (12) have three spatially homogeneous solutions: two stable solutions with the amplitudes

$$u = 0, \quad |u_{mi}|^2 = \frac{1}{2}\beta_i + \sqrt{\frac{1}{4}\beta_i^2 - (1 + k_{0i}^4)} \qquad (13)$$

for Eq. (11), and

$$u = 0, \quad u_{mi} = \frac{1}{2}\beta_i + \sqrt{\frac{1}{4}\beta_i^2 - (1 + k_{0i}^4)} \qquad (14)$$

for Eq. (12), and one unstable solution with the amplitude

$$|u_{ci}|^2 = \frac{1}{2}\beta_i - \sqrt{\frac{1}{4}\beta_i^2 - (1 + k_{oi}^4)} \qquad (15)$$

for model (11), and with the amplitude

$$u_{ci} = \frac{1}{2}\beta_i - \sqrt{\frac{1}{4}\beta_i^2 - (1 + k_{oi}^4)} \qquad (16)$$

for model (12). Besides, periodic, quasiperiodic, and stochastic distributions may correspond to these attractors. Indeed, all static solutions of Eqs. (11), (12) that are established as $t \to \infty$ meet the translational dynamical systems

$$\delta F_i^{(1)}/\delta u_i^* = 0 \qquad (17)$$

and

$$\delta F_i^{(2)}/\delta u_i = 0\,, \qquad (18)$$

respectively. In the phase space of these systems there exist, besides equilibrium states, periodic and quasiperiodic trajectories, and a homoclinic structure (see, for example, Pikovsky & Rabinovich [1984]). The latter indicates that chaotic spatial distributions of the field may emerge in the evolution problems (11)–(12) (see Rabinovich *et al.*, [1992], Gorshkov *et al.*, [to be published]). Taking into account $\pm v\partial_x$ (i.e., passing over to the moving coordinate system) we find that the "fixed" spatial distribution which is established as a result of the decrease of the free energy functional will drift towards the right or towards the left depending on the sign of the group velocity.

The behavior of models (1)–(2) and (3)–(4) at small couplings ($\varepsilon_i \ll 1$) is primarily the same as at $\varepsilon_i = 0$. In this case, however, the free energy functional that is descreasing in the course of pattern formation will not remain constant in time as soon as the process has been completed. Instead, it will fluctuate (periodically or stochastically) depending on the type of attractor. The spatial structures moving in opposite directions cease to be "frozen" and their interaction leads to nonstationary oscillations of the field in time. A new effect — competition of localized structures – must appear with increase of coupling.

The interaction of counterpropagating modes may be analysed by investigating elementary spatial distributions, i.e., localized structures of the field. In the phase space of the translationl dynamical system (17),(18), a closed homoclinic trajectory emerging from the coordinate origin and enveloping the equilibrium states with the amplitude $|u_{mi}|$ [see (13), (14)] corresponds to a stationary localized solution of Eqs. (11),(12) [Pikovsky & Rabinovich, 1984].

We can readily obtain the condition necessary for the existence of localized solutions for Eq. (12). By differentiating Eq. (18) with respect to x, multiplying by u_x, and integrating from $-\infty$ to $+\infty$ we obtain, after some transformations, the following expression,

$$-\int_{-\infty}^{+\infty} \{u_{ix}^2(1 - 2\beta_i u_i + 3u_i^2) + ((k_{0i}^2 + \partial_x^2)u_{ix})^2\} dx = 0\,. \qquad (19)$$

Apparently, the left-hand side of (19) is nonpositive for $\beta_i < \sqrt{3}$; consequently, $u_x \equiv 0$. Thus, the localized states that vanish to zero at infinity exist only for $\beta > \sqrt{3}$.

By virtue of the hard regime of excitation within Eqs.(11),(12) the necessary condition for the formation of a localized structure is that the amplitude of initial perturbation should exceed the threshold value determined by the amplitude of unstable spatial homogeneous solution $|u_{ci}|$ [see (15),(16)]. The width and amplitude of the structure formed do not depend on initial conditions but are determined by the parameters of the system.

Our first task is to analyse (numerically and analytically) the interaction of two solitary localized states of the fields u_1 and u_2 within the Ginzburg–Landau model (1)–(2) (the same considerations are valid for model (3)–(4)). The analysis gives an analytical estimate for the time of soliton interaction that depends on their width and group velocity and at which, when exceeded, the solitons may vanish.

We will describe each soliton as follows [Aranson *et al.*, 1990]:

$$u_i(x) = \begin{cases} u_{mi}, & |x - x_{0i}| \le r_{0i}, \\ u_{mi}\exp(-\mu_i|x - x_{0i}|) \\ \cos(\nu_i|x - x_{0i}|), \\ |x - x_{0i}| > r_{0i}, \end{cases} \quad (20)$$

where $|u_{mi}|$ is the amplitude of the ith soliton ($i = 1, 2$), x_{0i} is the coordinate of its center, r_{0i} is soliton's halfwidth, $\mu_i = |\mathrm{Re}\sqrt{i - k_{0i}^2}|$, and $\nu_i = |\mathrm{Im}\sqrt{i - k_{0i}^2}|$.

Knowing the structure of the phase space of the translational dynamical system (17) we may take as an amplitude $|u_{mi}|$ of the localized structure the amplitude of the stable nontrivial spatially homogeneous solution[1] (13).

Making use of (20) we can easily evaluate the stability region, $\beta_{1i}(k_{0i}) < \beta_i < \beta_{2i}(k_{0i})$, of localized solution on the parameter plane. To this end we minimize the free energy functional $F_i^{(1)}$ for solutions of the form (20). The final expression is too tedious to be presented here. Typical dependences $F_i^{(1)}(r_{0i})$ are given in Fig. 1 for different values of the parameters β_i and k_{0i}.

Apparently, the localized structure (20) is stable only if the function $F_i^{(1)}(r_{0i})$ has a minimum at some $r_{0i} = r_{0i}^* \neq 0$ (Fig. 1a). When $\beta_i < \beta_{1i}$, the function $F_i^{(1)}(r_{0i})$ has only one minimum for $r_{0i} = 0$ and grows monotonically as r_{0i} increases (Fig. 1b). Consequently, the system profits energetically if it changes to the state with $u_i \equiv 0$ when the localized structure (20) collapses. On the contrary, when $\beta_i > \beta_{2i}$, the function $F_i^{(1)}(r_{0i})$ decreases monotonically as $r_{0i} \to \infty$ (Fig. 1c). In this case, a stable nontrivial homogeneous state (13) is more profitable energetically.

An expression for the upper boundary of stability region may be written in an explicit form. It should be borne in mind that the contribution of exponentially small soliton "tails" may be neglected for sufficiently large r_{0i}. Then the expression for $F_i^{(1)}(r_{0i})$ takes on a form

[1]In fact, we consider a localized structure as a combination of two differences between stable spatially homogeneous solutions with the amplitudes $u \equiv 0$ and $|u| \equiv |u_{mi}|$.

$$\begin{aligned} F_i^{(1)}(r_{0i}) &= 2r_i|u_{mi}|^2\Big\{(1 + k_{0i}^4) \\ &\quad -\frac{1}{2}\beta_i|u_{mi}|^2 + \frac{1}{3}|u_{mi}|^4\Big\} \\ &= \frac{4}{3}r_{0i}|u_{mi}|^2\Big\{(1 + k_{0i}^4) - \frac{1}{4}\beta_i|u_{mi}|^2\Big\}. \end{aligned}$$

For $1/4\beta_i|u_{mi}|^2 > (1 + k_{0i}^4)$, the increase in r_{0i} leads to a decrease of $F_i^{(1)}(r_{0i})$ and, consequently, the system goes to a nonlocalized state. Using (13) we obtain an expression for β_{2i} [Aranson *et al.*, 1990]:

$$\beta_{2i} = (4/\sqrt{3})\sqrt{1 + k_{0i}^4}. \quad (21)$$

The region of stability of the localized solution (20) for model (12) may be estimated in a similar fashion. In particular, the upper boundary of the region is estimated as

$$\beta_{2i} = (2/\sqrt{3})\sqrt{1 + k_{0i}^4}. \quad (22)$$

The problem of interaction of localized states on counterpropagating modes will be solved employing the method of successive approximations. Assuming that solitons do not interact in the zero approximation, we can write for system (1)–(2) an expression in the first-order approximation:

$$\begin{aligned} \partial_t u_i \pm v\partial_x u_i &= -\frac{\delta F_i^{(1)}}{\delta u_i^*} \\ &\quad -\varepsilon_i u_i|u_j(x \pm vt)|^2, \\ i &= 1, 2, \end{aligned} \quad (23)$$

where $u_j(\xi)$ in the last term is given by expression (20). This term is responsible for the influence of counterpropagating wave on the given localized structure. Clearly, with the field of the counterpropagating mode being localized, the localization region contributes significantly to interaction.

Below we will assume that there is a rather strong difference between the parameters of equations for the counterpropagating modes ($\beta_i \ll \beta_j$). Then $\Delta_i \ll \Delta_j$ and it can be assumed that the soliton of the ith mode is entirely in the localization region of the soliton of the jth mode throughout interaction. Neglecting the effect of the "tails" of solution $u_j(x)$ that is exponentially small as compared with the contribution of the center of the soliton and restricting ourselves to a consideration of the time interval during which the solitons interact, we can rewrite (23) in the form

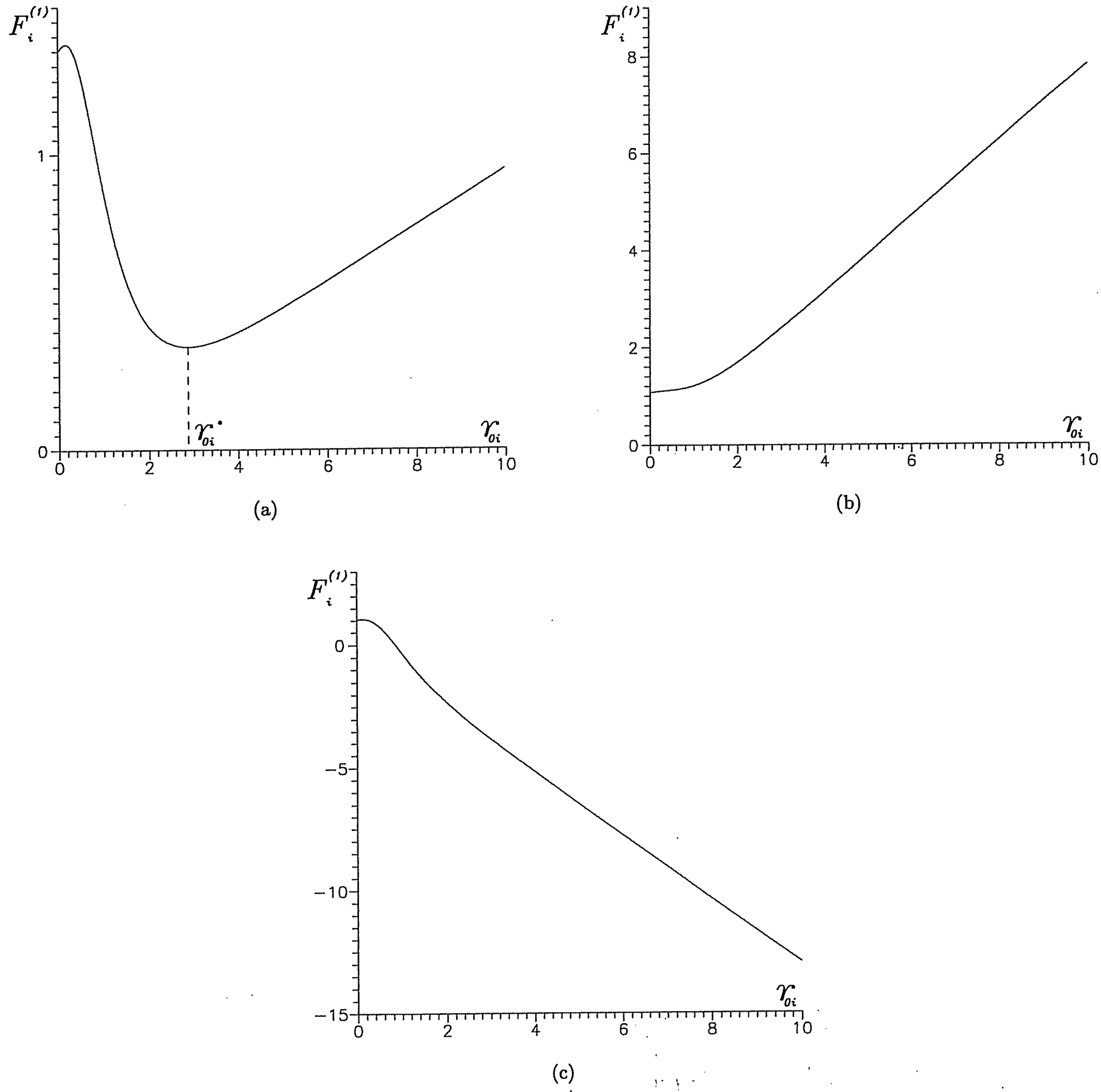


Fig. 1. Dependence of the free energy functional (5) on soliton radius r_{0i} for $k_{0i} = 0.5$: (a) $\beta_{0i} = 2.35$, the localized structure is stable; (b) $\beta_{0i} = 2.0$, the localized structure collapses and a trivial spatially homogeneous regime is established ($u_i \equiv 0$); (c) $\beta_{0i} = 2.7$, the localized structure spreads and a spatially homogeneous regime with a zero field is established.

$$\partial_t u_i \pm v\partial_x u_i = -\frac{\delta \overline{F}_i^{(1)}}{\delta u_i^*}\,, \tag{24}$$

where

$$\overline{F}_i^{(1)} = \int \{|u_i|^2(1 + \varepsilon_i|u_{mj}|^2 - \frac{1}{2}\beta_i|u_i|^2 + \frac{1}{3}|u_i|^4) + |(k_{0i}^2 + \partial_x^2)u_i|^2\}dx\,.$$

Equation (24) contains an additional term $-\varepsilon_i u_i|u_{mj}|^2$ [cf. Eq. (11)]. Clearly, stable localized solutions of Eq. (24) will exist at $\varepsilon_i \neq 0$, given that additional damping due to this term is compensated by the increase of nonlinear increment, which, in turn, shifts the boundary of the stability region of localized solutions towards larger β.

For example, for $k_{0i} = 0$ in the absense of coupling between the counterpropagating modes ($\varepsilon_i = 0$), the boundary of stability region of the solutions to (20) that is found from the conditions of the existence of a nontrivial minimum of free energy functional $\overline{F}_i^{(1)}$ is determined by $\beta > \beta_{1i} \simeq 2.07$. While in the presence of coupling ($\varepsilon_i = 0.1$) and with $|u_{mi}|^2 \simeq 1.75$ (the amplitude of the soliton of the counterpropagating mode for $\beta_j = 2.35$, $k_{0i} = 0.5$), the stability condition takes on the form $\beta_i > \overline{\beta}_{1i} \simeq 2.2$.

Thus, while at $\varepsilon_i = 0$ the parameters of the equation belong to the region of stable localized solutions, they may move to the instability region when $\varepsilon_i \neq 0$ and the amplitude of the localized structure will begin to decrease.

Regarding the localized structure (20) as a domain of an excited state of the field and neglecting boundary effects (the effect of oscillating "tails"), the velocity with which the amplitude of localized state changes may be estimated from the equation

$$\frac{du_i}{dt} = -(1 + k_{0i}^4 + \varepsilon_i|u_{mj}|^2)u_i + \beta_i u_i|u_i|^2 - u_i|u_i|^4\,, \quad i \neq j\,, \tag{25}$$

with the boundary conditions $|u_i(0)| = |u_{mi}|$.

Now the problem of soliton "survival" reduces to whether the soliton amplitude attains during interaction the critical value $|u_{ci}|$, i.e., the amplitude of the unstable spatially homogeneous solution of Eq. (11) [see (15)].

Rewriting Eq. (25) for squared absolute value of the field amplitude and performing integration we find a critical value for the time of soliton interaction:

$$t_{\text{cri}} = \frac{1}{2}\int_{|u_{mi}|^2}^{|u_{ci}|^2} \times \frac{d|u_i|^2}{-(1+k_{0i}^4+\varepsilon_i|u_{mj}|^2)|u_i|^2+\beta_i|u_i|^4-|u_i|^6}\,, \tag{26}$$

and an expression for the critical velocity of relative motion of the solitons:

$$v_{\text{cri}} = \Delta_j/t_{\text{cri}}\,. \tag{27}$$

We would like to remind our readers that in the derivation of formulas (26) and (27) we assume that the condition $\beta_i \ll \beta_j$ is met, which allows for the two following approximations. First, we neglect interaction of the solitons when they move one against another and overlap only partially. Given $\Delta_i \ll \Delta_j$, these time intervals are short as compared with the time during which one of the solitons is located entirely in the localization region of the other. Second, it is assumed that the jth soliton's amplitude that enters the right-hand side of the equation for the ith soliton is constant during interaction. Actually this is not the case because the amplitudes of both solitons decrease. Therefore the theoretical value of v_{cri} found from (26) and (27) must be overestimated at close values of the parameters ($\beta_i \approx \beta_j$). Alternatively, for a rather great difference in the parameters ($\beta_i \ll \beta_j$), the amplitude of the ith soliton will decrease in the course of interaction faster than that of the jth soliton and the error in (26), (27) due to constant amplitude of the jth soliton will be small. It should also be emphasized that for values of β_i close to the lower boundary of stability region (for a narrow soliton), the inaccuracy of approximating a rectangular shape of the soliton increases and the oscillating tails of the solution to (20) must be taken into account to make a more adequate description.

Figure 2 shows the critical value of group velocity $v_{\text{cr}2}$ plotted versus the parameter β_2 at constant $\beta_1 = 2.35$ that was obtained in computer calculations[2] (solid curve) and a similar dependence

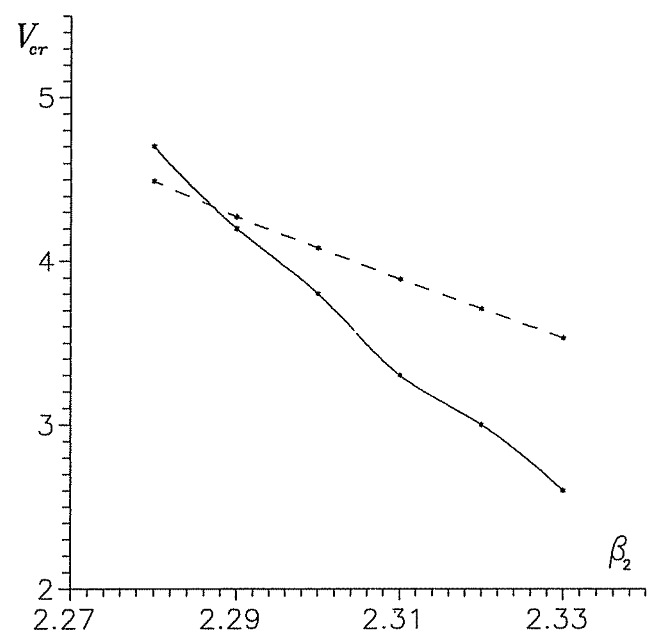


Fig. 2. Critical velocity v_{cr} as a function of the parameter β_2 for $k_{01} = k_{02} = 0.5$, $\beta_1 = 2.35$. The solid curve is calculated numerically and the broken line is found analytically.

[2]The boundary value problems (1)–(2) and (3)–(4) were solved on a computer employing a combination of a pseudospectral method [Orszag, 1971] and a method of operator exponent [Aranson *et al.*,1991].

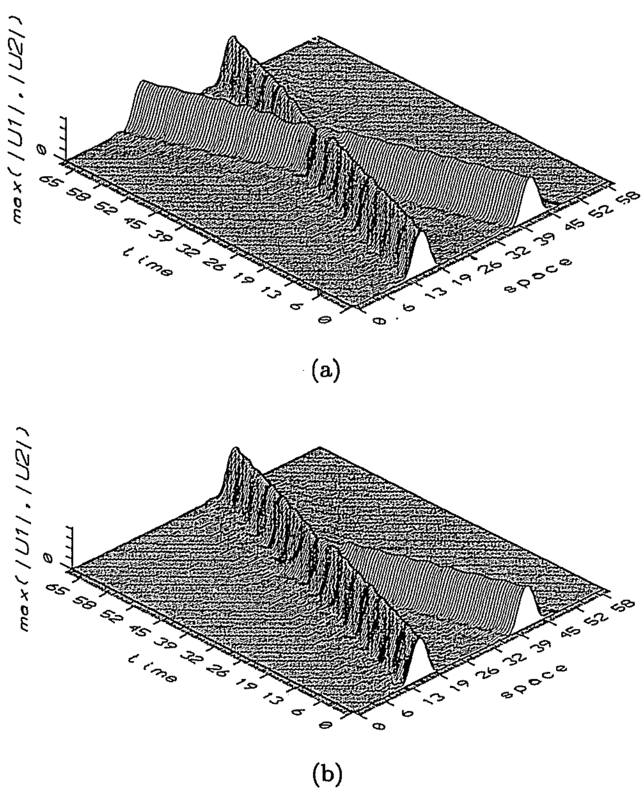


Fig. 3. Time evolution of two solitons of the form (1)–(2) for $k_{01} = k_{02} = 0.5$, $\beta_1 = 2.35$: (a)$\beta_2 = 2.35$, both solitons reconstruct their shape on completion of interaction; (b) $\beta_2 = 2.3$, one of the solitons disappears as a result of interaction.

found from expressions (26)–(27) (broken curve). One can see that the difference between numerical and theoretical values of $v_{\text{cr}2}$ grows with the increase of the parameter β_2 and attains its maximum when $\beta_1 \approx \beta_2$.

Time evolution of solitons within model (1)–(2) is depicted in Fig. 3 for different ratios of the parameters β_1 and β_2. In the first case ($\beta_1 = \beta_2 = 2.35$) the shape of the solitons after they have passed through one another is reconstructed completely. While for $\beta_1 = 2.35$, $\beta_2 = 2.30$, the second soliton vanishes.

3. Interaction of Lattices

We have already mentioned that the localized structures considered above may form a lattice of structures. This distribution is steady state in the absence of coupling ($\varepsilon_i = 0$) and corresponds to one of the minima of free energy functional. The solitons forming the lattice may be arranged either regularly or randomly, depending on the initial conditions.

In the presence of coupling ($\varepsilon_i \neq 0$), the soliton lattices moving towards one another interact. If the distance between solitons in the lattice is much greater than the typical length of the soliton, the type of interaction is determined only by the interaction of individual solitons, as was described above, because the soliton is fully reconstructed in its motion between collisions.

Soliton interaction acquires new features if the distance between solitons in the lattice is of the order of their characteristic size. Then, the soliton may fail to reconstruct its amplitude during motion between collisions. Apparently, the soliton amplitude will drop below its threshold amplitude after several collisions and the soliton will vanish even if the group velocity of the lattice is greater than the critical velocity found theoretically for individual solitons.

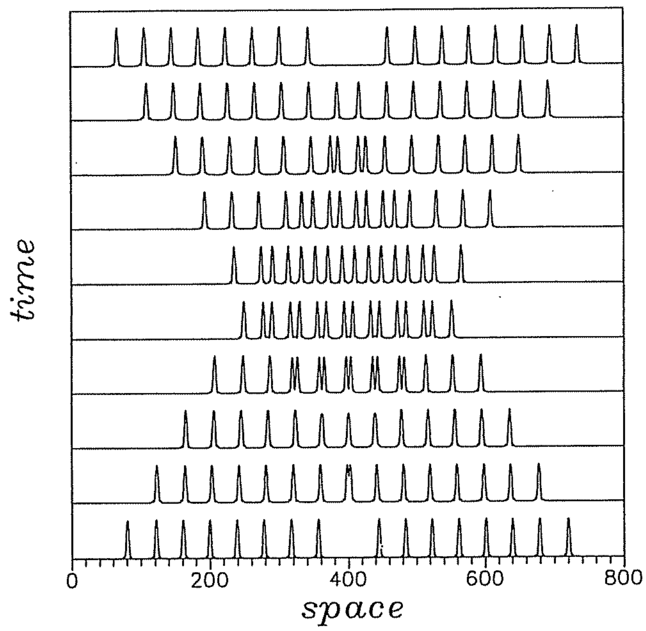
A situation that is, in a sense, inverse to the one described above is also possible. If the parameters β_1, β_2 of Eqs. (1)–(2) or Eqs. (3)–(4) exceed the upper boundary of stability region [see (21), (22)], then the lattices will be unstable in the absence of interaction and, in some time, a nontrivial spatially homogeneous state of the field will be established in each system. The moving lattices may "stabilize" one another in the presence of coupling ($\varepsilon_i \neq 0$). Soliton spreading in time intervals when they do not interact with counterpropagating solitons is compensated by the decrease of their amplitude during interaction and, as a whole, the solitons retain their shape.

The snapshots of two interacting lattices of model (1)–(2) are shown in Fig. 4 at different moments of time. One can see that depending on the parameter ratio in the equations one of the following situations may be realized:

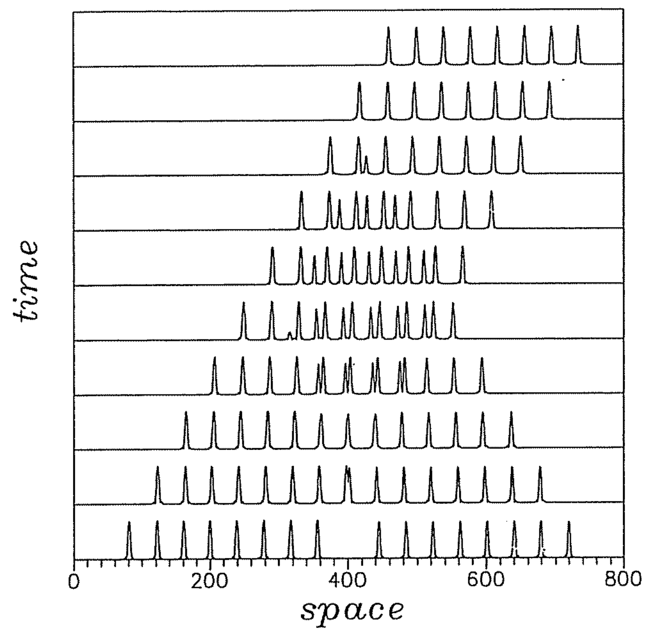
- both lattices fully retain their structure in the process of interaction and reconstruct their shape on its completion (Fig. 4a);
- one of the lattices disappears but each soliton retains its shape in the course of several collisions (Fig. 4b);
- each soliton of the second lattice vanishes after the first collision with the counterpropagating lattice (Fig. 4c);
- finally, both the lattices disappear as a result of the interaction (Fig. 4d).

4. Quasiconservative Model

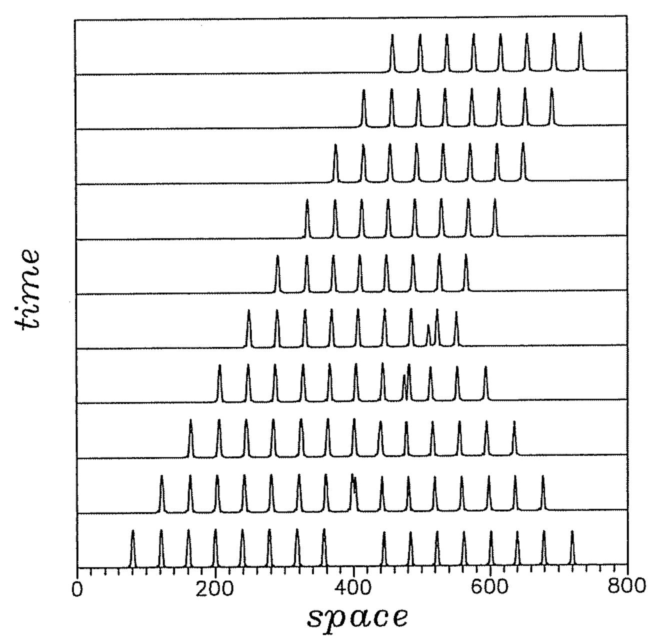
Now consider the interaction of localized structures of a nonlinear field within two coupled Ginzburg–Landau equations with complex coefficients (5)–(6).



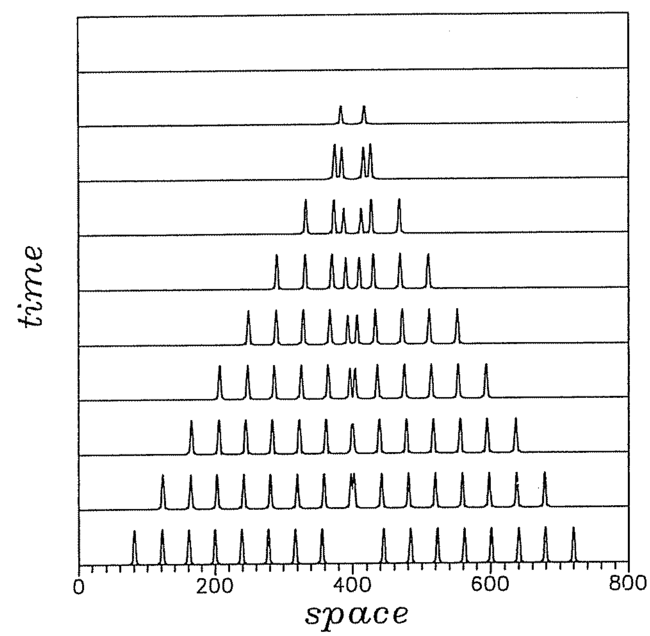
(a)



(b)



(c)



(d)

Fig. 4. Time evolution of two periodic soliton lattices of the form (1)–(2) for $k_{01} = k_{02} = 0.5$: (a) $\beta_1 = \beta_2 = 2.35$, both lattices reconstruct their shape on completion of interaction; (b) $\beta_1 = 2.35$, $\beta_2 = 2.288$, one of the lattices disappears after several collisions with counterpropagating mode solitons; (c) $\beta_1 = 2.35$, $\beta_2 = 2.25$, one of the lattices disappears after the first collision with counterpropagating mode solitons; (d) $\beta_1 = \beta_2 = 2.25$, both lattices disappear as a result of interaction.

First of all we will investigate in the linear approximation the stability of spatially homogeneous solutions of system (5)–(6) in the form

$$u_i = A_i \exp(-i\omega_i t)\,, \tag{28}$$

where

$$A_i^2 = \frac{1 - l_i}{1 - l_i l_j}\,, \quad \omega_i = \beta_i A_i^2 + d_i A_i^2\,.$$

The perturbed solution may be written as

$$u_i(x,t) = (A_i + s_i(x,t))\exp(-i\omega_i t)\,. \qquad (29)$$

Substituting (29) into (5)–(6) and retaining only linear terms we find ($v = 0$)

$$\begin{aligned}\frac{\partial s_i}{\partial t} &= (1+i\omega_i)s_i - (1+i\beta_i)(2s_i + s_i^*)\\ &\quad +(1+i\alpha)\frac{\partial^2 s_i}{\partial x^2}\\ &\quad -(l_i + id_i)[(s_i + s_j^*)A_iA_j + s_iA_j^2]\,,\\ i &= 1,2\,. \end{aligned} \qquad (30)$$

We take interest in the stability of the equilibrium solution $s_i = 0$ of system (30) with respect to perturbations of the form

$$s_i \sim \exp(ik_n x)\,, \quad k_n = 2\pi n/L\,, \qquad (31)$$

where L is the resonator length.

For the sake of simplicity, we assume that $l_1 = l_2 = l$ and $d_1 = d_2 = d$. We can readily show that, in this case, the Lyapunov exponents of perturbations (31) meet the equation

$$\begin{aligned}&\lambda^4 + 2\lambda^3(2k_n^2 + h) + \lambda^2[(2k_n^2+h)^2 + \rho_1 + \rho_2 - l^2h^2]\\ &\quad +\lambda[(2k_n^2+h)(\rho_1+\rho_2) - 2k_n^2l^2h^2 - ldh^2(\alpha_1\\ &\quad +\alpha_2)k_n^2] + [\rho_1\rho_2 - k_n^1 l^2h^2 - ldh^2k_n^4(\alpha_1+\alpha_2)\\ &\quad -h^2d^2(\alpha_1+\alpha_2)k_n^4] = 0\,,\end{aligned} \qquad (32)$$

where $\rho_i = hk_n^2(1+\alpha_i\beta_i) + k_n^4(1+\alpha_i)^2$, $h = 2/(1+l)$.

For a resonator of infinite length ($L \to \infty$), the stability condition may be written as follows:

$$\begin{aligned}&(1+\alpha_1\beta_1)(1+\alpha_2\beta_2) > l^2 + ld(\alpha_1+\alpha_2) + d^2\alpha_1\alpha_2\,,\\ &(1+\alpha_1\beta_1) + (1+\alpha_2\beta_2) > 2l^2 - ld(\alpha_1+\alpha_2)\,,\\ &l < 1\,.\end{aligned} \qquad (33)$$

When the conditions (33) are met, the real parts of all roots of Eq. (32) are negative and, consequently, the solutions (28) are linearly stable. It is readily seen that, in the absence of coupling ($l = d = 0$), the conditions (33) give the known Benjamin–Feir condition $1 + \alpha\beta > 0$ [Benjamin & Feir, 1966; Newell, 1974; Stuart & DiPrima, 1978].

We take in Eqs. (5)–(6) $l = d = 0$ and replace the variables $\xi_i = x \pm vt$. Then, we obtain one Ginzburg–Landau equation

$$\begin{aligned}\partial_t u_i &= u_i + (1+i\alpha_i)\partial_{\xi_i}^2 u_i\\ &\quad -(1+i\beta_i)u_i|u_i|^2\,.\end{aligned} \qquad (34)$$

Equation (34) has an exact localized solution of the form

$$u_i = u_{i0}(\sinh\lambda_i\xi_i)^{1+i\kappa_i}\exp(-i\Omega_i t)\,, \qquad (35)$$

where

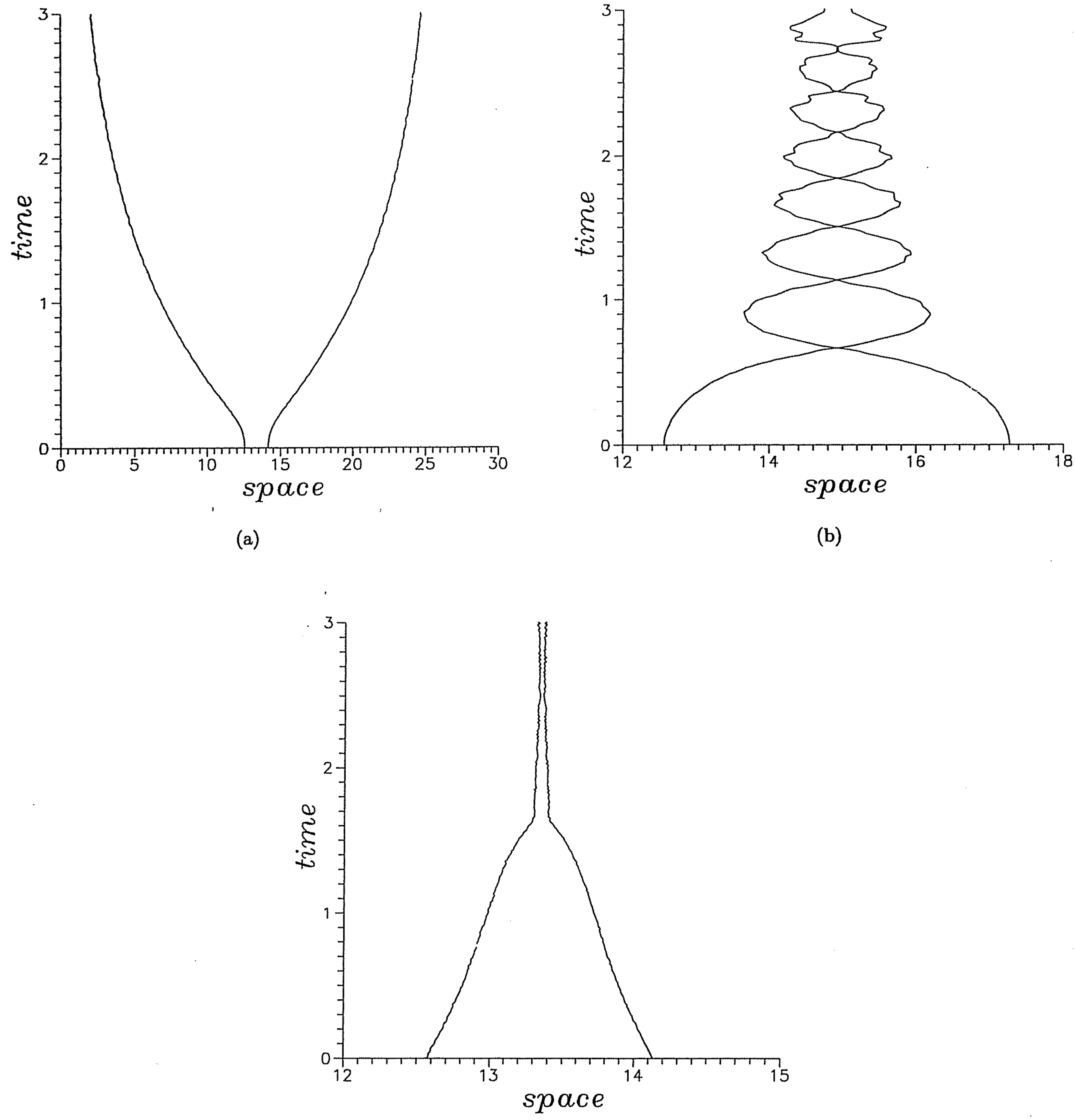
$$\begin{aligned}\kappa_i &= -\rho_i + (\rho_i^2 + 2)^{1/2}\,,\\ \rho_i &= -3(1+\alpha_i\beta_i)/2(\alpha_i - \beta_i)\,,\\ |u_{0i}|^2 &= \lambda_i^2(\kappa_i^2 - 2 + 3\kappa_i\alpha_i)\,,\\ \lambda_i &= 1/(\kappa_i^2 - 1 + 1\kappa_i\alpha_i)\,,\\ \Omega_i &= \lambda_i^2(\alpha_i(\kappa_i^2 - 1) - 1\kappa_i)\,.\end{aligned}$$

This solution was first found by Pereira and Stenflo [1977] and later by Nozaki and Bekki [1984] in the form presented above.

The interaction of localized structures at one mode [solutions (35) within one equation (34)] was investigated by Kishiba *et al.* [1991] in a nearly conservative case ($|\alpha_i|$, $|\beta_i| \gg 1$). It was shown that the localized solutions (35) may either repel or attract one another, depending on the ratio of initial phases. In the latter case the solitons pass through one another thus reconstructing their shape after the interaction.

In this paper we analyse the interaction of solitons of the form (35) at different modes, i.e., the ones described by two coupled Ginzburg–Landau equations (5)–(6). The parameters of the equations correspond to the quasiconservative case ($|\alpha_i|$, $|\beta_i| \gg 1$). We would like to note that, because we choose the parameters in the region where spatially homogeneous regime is unstable [see (33)], we can obtain a solitary solution of the form (34) only in a finite, although rather broad, time interval ΔT. As soon as this time elapses, a family of chaotically interacting solitons of the form (35) is formed in the system.

In our computer experiment, initially motionless solitons ($v = 0$) were arranged at a certain fixed distance from one another and their evolution



(a) (b) (c)

Fig. 5. Time evolution of the centers of solitons of the form (5)-(6) for $\alpha_1 = \alpha_2 = 100$, $\beta_1 = \beta_2 = -100$: (a) $l_1 = l_2 = 0$, $d_1 = d_2 = 1$, the solitons move apart; (b) $l_1 = l_2 = 0$, $d_1 = d_2 = -1$, the solitons move towards one another, oscillating relative to the common center; (c) $l_1 = l_2 = -1$, $d_1 = d_2 = 0$, the solitons monotonically move towards one another.

was studied in the time interval ΔT. We found that the character of soliton interaction depends significantly on the signs of the coefficients l_i, d_i. Two solitons repel if $l_i, d_i > 0$ (Fig. 5a) and, vice versa, they attract one another if $l_i, d_i < 0$ (Fig. 5b,c). The behavior of the solutions also depends on which coupling — real or imaginary[3] prevails. In the case of imaginary coupling, the solitons behave like

[3]Because of amplitude coupling of the equations in system (5)–(6) the type of interaction of localized solutions does not depend on their phase ratio.

particles in the potential of a given profile. The trajectories of two attracting solitons are shown in Fig. 5b for $l_i = 0$, $d_i = -1$. The solitons recurrently pass through one another oscillating near their common center. In the case of real coupling ($d_i = 0$), solitons do not oscillate and the distance between their centers decreases monotonically (Fig. 5c). The latter is explained by the fact that we are concerned with a nearly conservative system and the equations contain, primarily, imaginary coefficients.

5. Dimension of the Coupled Chaotic Lattices

The spatial field distributions that are established in the absence of coupling between the equations of models (1)–(2), (3)–(4), and (5)–(6) may be considered, at a certain moment of time t, as a result of the evolution of a finite-dimensional translational dynamical system G_x. Then we can introduce the correlation dimension of space series, d_s, and the Kolmogorov–Sinai entropy, h_s, which was first done by Afraimovich *et al.* [in press]. On the other hand, the spatial distribution that is moving at a certain group velocity at a given (fixed) point will generate a time series with correlation dimension d_t and entropy h_t. There arises a question: What is the relation between the spatial dimensions, d_{s1} and d_{s2}, and the dimension of the time series, d_t, for coupled subsystems?

We will investigate this problem starting from an analysis of the quasigradient models (1)–(2) and (3)–(4). Because the field distributions that are established, in the absence of coupling, at each counterpropagating mode are "frozen" for Eqs. (1)–(2) and (3)–(4), the time series for each mode will exactly repeat the space series and they will have equal characteristics (dimensions and entropies).

In the presence of coupling ($\varepsilon_i \neq 0$), each equation of system (1)–(2) or (3)–(4) may be considered, to the first approximation, as an equation with a variable parameter. Here it is a squared amplitude of the external (with respect to the given equation) field. The time dependence of this parameter is a sequence of pulses for which the repetition rate is determined by the second equation. Since the state of the system without coupling ($\varepsilon_i = 0$) corresponds to one of the minima of free energy functional, small coupling will produce the following effect: When the external parameter reaches at a given point of the field its maximal value (the "top" of the soliton), the nontrivial state of the field of the first equation at this point is unstable and its amplitude starts to decrease. At the next moment of time, the amplitude of the external parameter drops to zero (the field between the solitons) and the system tends to the initial state that corresponds to the minimum of the functional. Thus, the external parameter (i.e., the field amplitude of the second equation) modulates the field amplitude of the first equation. The dimension d_t of the time series that is generated within the first equation at a fixed point can be represented as a sum of the dimensions of spatial distribution inherent in this system and of external signal (spatial distribution of the second equation).

The considerations presented above are not quite exact because they do not take into account the feedback in the system ($\varepsilon_i \neq 0$ in both equations). However, if the coupling is small it is natural to assume that the following relation will hold:

$$d_t \simeq d_{s1} + d_{s2} \,. \tag{36}$$

As the coupling increases, the interaction of counterpropagating modes gradually "loosens" the equilibrium state corresponding to the fixed lattices. The system becomes significantly nonlinear far from the maxima of the free energy functional, its dynamics gets more and more complicated, and the dimension of the generated time series grows.

Results of computer experiments verify our theoretical suppositions. Numerical calculations show that the dimension of the time series, d_t, can be represented approximately as a sum of spatial characteristics of unperturbed static soliton lattices at $\varepsilon \ll 1$ and grows substantially as ε increases (see Fig. 6). With a further increase of coupling, the dynamics of the system changes qualitatively. Above the critical value $\varepsilon_{\rm cr}$, individual localized structures and groups of structures at one of the modes (the choice of the mode depending on initial conditions) are suppressed by the counterpropagating mode. Eventually, a spatially homogeneous regime is established at this mode.

Now consider the relation between the dimension of the time series measured at a fixed point and the dimension of the space series for counterpropagating stochastic fields of the quasi-conservative model (5)–(6). A nonstationary field distribution is established for each traveling wave in system (5)–(6), i.e. the field distribution varies in time even when the subsystems are not coupled ($l_i = d_i = 0$). This indicates that for each wave the dimension of the time series, d_{ti}, measured at a fixed point relative to which the wave is traveling at a speed v, depends on this speed and does not

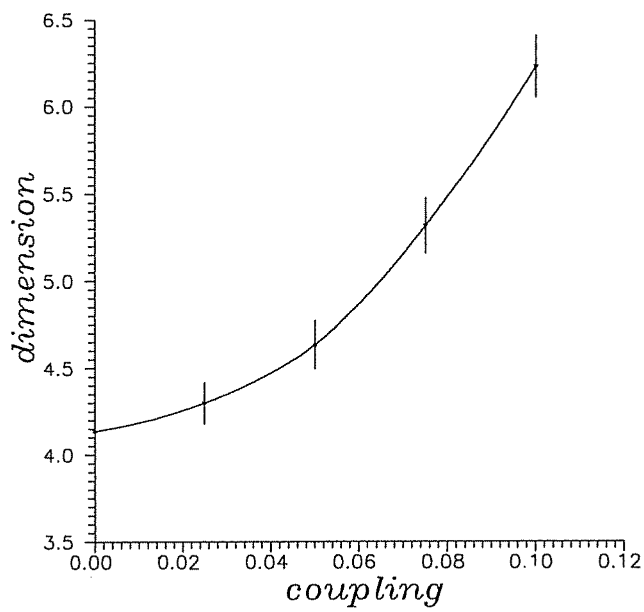


Fig. 6. The dimension of the time series, d_t, as a function of the coupling between modes, ε, for model (3)-(4) : $k_{01} = k_{02} = 0.5$, $\beta_1 = \beta_2 = 2.18$.

coincide with the dimension of the space series, d_{si}. Consequently, the dimension of the time series, $d_t = d_{t1} + d_{t2}$, differs from the dimension of the space series, $d_s = d_{s1} + d_{s2}$, measured at a certain moment of time t even when two counterpropagating waves are not coupled in the system. For example, for $\alpha_i = -0.9$, $\beta_i = 1.2$, $l_i = d_i = 0$, and $v = \pm 0.2$ we have $d_{s1} = d_{s2} = 4.4 \pm 0.1$ while $d_t = 10.4 \pm 0.2$.

Thus, unlike quasigradient systems, there is no apparent connection between spatial and temporal dimensions in a nongradient case. The relation between spatial and temporal dimensions within nongradient systems is still open for discussion.

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